

# On Irreducibility of Tensor Products of Yangian Modules

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## Introduction

In the present article we continue our study [NT] of the finite-dimensional modules over the Yangian  $Y(\mathfrak{gl}_N)$  of the general linear Lie algebra  $\mathfrak{gl}_N$ . The Yangian  $Y(\mathfrak{gl}_N)$  is a canonical deformation of the universal enveloping algebra  $U(\mathfrak{gl}_N[u])$  in the class of Hopf algebras [D1]. Definition of the algebra  $Y(\mathfrak{gl}_N)$  in terms of an infinite family of generators  $T_{ij}^{(s)}$  with  $s = 1, 2, \dots$  and  $i, j = 1, \dots, N$  is given by (1.1), (1.2). Comultiplication  $\Delta : Y(\mathfrak{gl}_N) \rightarrow Y(\mathfrak{gl}_N) \otimes Y(\mathfrak{gl}_N)$  is defined by (1.1), (1.16).

The algebra  $Y(\mathfrak{gl}_N)$  admits an alternative definition in terms of the ascending chain  $U(\mathfrak{gl}_1) \subset U(\mathfrak{gl}_2) \subset \dots$  of classical universal enveloping algebras [O]. For any non-negative integer  $M$  consider the commutant in  $U(\mathfrak{gl}_{M+N})$  of the subalgebra  $U(\mathfrak{gl}_M)$ . This commutant is generated by the centre of the subalgebra  $U(\mathfrak{gl}_M)$  and a homomorphic image of the Yangian  $Y(\mathfrak{gl}_N)$ , see Proposition 1.1. The intersection of the kernels in  $Y(\mathfrak{gl}_N)$  of all these homomorphisms for  $M = 0, 1, 2, \dots$  is zero.

For any dominant integral weights  $\lambda$  and  $\mu$  of the Lie algebras  $\mathfrak{gl}_{M+N}$  and  $\mathfrak{gl}_M$  consider the subspace  $V_{\lambda, \mu}$  in the irreducible  $\mathfrak{gl}_{M+N}$ -module  $V_\lambda$  of highest weight  $\lambda$  formed by all singular vectors with respect to  $\mathfrak{gl}_M$  of weight  $\mu$ . The algebra  $Y(\mathfrak{gl}_N)$  acts in  $V_{\lambda, \mu}$  irreducibly through the above homomorphism. Further, for any complex number  $h$  there is an automorphism  $\tau_h$  of the algebra  $Y(\mathfrak{gl}_N)$  defined in terms of the generating series (1.1) by the assignment  $T_{ij}(u) \mapsto T_{ij}(u + h)$ . By pulling back the  $Y(\mathfrak{gl}_N)$ -module  $V_{\lambda, \mu}$  through this automorphism we obtain an irreducible  $Y(\mathfrak{gl}_N)$ -module, which we denote by  $V_{\lambda, \mu}(h)$  and call elementary.

Any formal series  $f(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$  also defines an automorphism  $\omega_f$  of the algebra  $Y(\mathfrak{gl}_N)$  by  $T_{ij}(u) \mapsto f(u) \cdot T_{ij}(u)$ . Further, there is a canonical chain of algebras  $Y(\mathfrak{gl}_1) \subset \dots \subset Y(\mathfrak{gl}_N)$ . The elementary modules are distinguished amongst all irreducible finite-dimensional  $Y(\mathfrak{gl}_N)$ -modules  $W$  by the following fact. Take the commutative subalgebra  $A(\mathfrak{gl}_N)$  in  $Y(\mathfrak{gl}_N)$  generated by the centres of all algebras in the latter chain. Then the action of this subalgebra in  $W$  is semi-simple, if and only if  $W$  is obtained by pulling back through some automorphism  $\omega_f$  from a tensor product of elementary  $Y(\mathfrak{gl}_N)$ -modules

$$V = V_{\lambda^{(1)}, \mu^{(1)}}(h^{(1)}) \otimes \dots \otimes V_{\lambda^{(n)}, \mu^{(n)}}(h^{(n)})$$

where  $h^{(r)} - h^{(s)} \notin \mathbb{Z}$  whenever  $1 \leq r < s \leq n$ . This fact was conjectured in [C2] and proved in [NT]. It was subsequently applied in [KKN] and [TU] to the analysis of integrable lattice models. If  $h^{(r)} - h^{(s)} \notin \mathbb{Z}$  for all  $r < s$  then  $Y(\mathfrak{gl}_N)$ -module  $V$  is irreducible and the action of  $A(\mathfrak{gl}_N)$  in  $V$  has a simple spectrum. Here we study the tensor product  $V$  when the differences  $h^{(r)} - h^{(s)}$  are any complex numbers.

Theorem 3.3 gives sufficient conditions for irreducibility of  $Y(\mathfrak{gl}_N)$ -module  $V$ . In general, these conditions are not necessary for the irreducibility of  $V$ . However, in the particular case when  $\lambda^{(1)}, \dots, \lambda^{(n)}$  are any fundamental weights of  $\mathfrak{gl}_N$  whilst  $\mu^{(1)}, \dots, \mu^{(n)}$  are empty, the conditions of Theorem 3.3 are necessary for the irreducibility of  $V$  as well. This follows from Theorem 3.2. In this particular

case the irreducibility criterion of  $V$  was obtained in [AK] by using the technique of crystal bases. Our approach is different. We use an eigenbasis for the action of  $A(\mathfrak{gl}_N)$  in each tensor factor  $V_{\lambda^{(s)}, \mu^{(s)}}(h^{(s)})$  of  $V$ . This eigenbasis is a part of the Gelfand-Zetlin basis [GZ] in the  $\mathfrak{gl}_{M^{(s)}+N}$ -module  $V_{\lambda^{(s)}}$  corresponding to the chain of Lie algebras  $\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \dots \subset \mathfrak{gl}_{M^{(s)}+N}$ . Note that a vector  $\zeta^{(s)} \in V^{(s)}$  called singular is contained in this eigenbasis.

Recall the notion of a universal  $R$ -matrix [D1] for the Hopf algebra  $Y(\mathfrak{gl}_N)$ . Let  $\Delta'$  be the composition of the comultiplication  $\Delta$  with permutation of the tensor factors in  $Y(\mathfrak{gl}_N) \otimes Y(\mathfrak{gl}_N)$ . There exists a formal power series  $\mathcal{R}(z)$  in  $z^{-1}$  with coefficients from  $Y(\mathfrak{gl}_N) \otimes Y(\mathfrak{gl}_N)$  and the leading term 1 such that

$$\mathcal{R}(z) \cdot \text{id} \otimes \tau_z(\Delta'(y)) = \text{id} \otimes \tau_z(\Delta(y)) \cdot \mathcal{R}(z)$$

for all  $y \in Y(\mathfrak{gl}_N)$ . For any  $r < s$  there is a formal series  $f(z) \in 1 + z^{-1}\mathbb{C}[[z^{-1}]]$  such that  $f(z)\mathcal{R}(z)$  acts in  $V_{\lambda^{(r)}, \mu^{(r)}}(h^{(r)}) \otimes V_{\lambda^{(s)}, \mu^{(s)}}(h^{(s)})$  as a rational function in  $z$ . This rational function may have zero or pole at  $z = 0$ . Taking the first non-zero coefficient of the Laurent series of this function at  $z = 0$  we get an element

$$R^{(rs)} \in \text{End}(V_{\lambda^{(r)}, \mu^{(r)}} \otimes V_{\lambda^{(s)}, \mu^{(s)}})$$

which is an intertwining operator between the  $Y(\mathfrak{gl}_N)$ -modules obtained from the tensor product  $V_{\lambda^{(r)}, \mu^{(r)}}(h^{(r)}) \otimes V_{\lambda^{(s)}, \mu^{(s)}}(h^{(s)})$  via comultiplications  $\Delta'$  and  $\Delta$ . This intertwining operator is made explicit in Proposition 2.2, it depends on the parameters  $h^{(r)}$  and  $h^{(s)}$  via their difference  $h^{(r)} - h^{(s)}$ . If this operator is non-invertible for some  $r < s$  then the  $Y(\mathfrak{gl}_N)$ -module  $V$  is reducible.

**Conjecture.** *The  $Y(\mathfrak{gl}_N)$ -module  $V$  is irreducible if and only if all the operators  $R^{(rs)}$  with  $1 \leq r < s \leq n$  are invertible.*

Our Theorem 3.4 confirms this conjecture when  $\lambda^{(1)}, \dots, \lambda^{(n)}$  are multiples of any fundamental weights of  $\mathfrak{gl}_N$  while  $\mu^{(1)}, \dots, \mu^{(n)}$  are empty. Then Theorem 2.3 describes explicitly the set of differences  $h^{(r)} - h^{(s)} \in \mathbb{Z}$  where the operator  $R^{(rs)}$  is not invertible, see the remarks after the proof of Theorem 3.4. The proofs of Theorems 3.3 and 3.4 are based on Proposition 3.1. It gives sufficient conditions on the parameters  $h^{(1)}, \dots, h^{(n)}$  for cyclicity of the vector  $\zeta^{(1)} \otimes \dots \otimes \zeta^{(n)}$  in the  $Y(\mathfrak{gl}_N)$ -module  $V$  with arbitrary  $\lambda^{(1)}, \dots, \lambda^{(n)}$  and  $\mu^{(1)}, \dots, \mu^{(n)}$ .

## 1. Elementary modules

In this section we consider a class of irreducible finite-dimensional  $Y(\mathfrak{gl}_N)$ -modules which arise naturally from the classical representation theory [C2,O]. Here we also collect all necessary facts from [MNO,NT] about the Hopf algebra  $Y(\mathfrak{gl}_N)$ .

The *Yangian* of general linear Lie algebra  $\mathfrak{gl}_N$  is the associative unital algebra  $Y(\mathfrak{gl}_N)$  over  $\mathbb{C}$  with the generators  $T_{ij}^{(s)}$  where  $s = 1, 2, \dots$  and  $i, j = 1, \dots, N$ . Defining relations in the algebra  $Y(\mathfrak{gl}_N)$  can be written for the generating series

$$(1.1) \quad T_{ij}(u) = \delta_{ij} + T_{ij}^{(1)}u^{-1} + T_{ij}^{(2)}u^{-2} + \dots$$

in a formal parameter  $u$  as follows: for all indices  $i, j, k, l = 1, \dots, N$  we have

$$(1.2) \quad (u - v) \cdot [T_{ij}(u), T_{kl}(v)] = T_{kj}(v)T_{il}(u) - T_{kj}(u)T_{il}(v).$$

Here  $v$  is another formal parameter. Let us rewrite these relations in a matrix form

Let  $E_{ij} \in \text{End}(\mathbb{C}^N)$  be the standard matrix units. Combine all the series  $T_{ij}(u)$  into the single element

$$T(u) = \sum_{i,j=1}^N E_{ij} \otimes T_{ij}(u)$$

of the algebra  $\text{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N)[[u^{-1}]]$ . Consider the *Yang R-matrix*

$$(1.3) \quad R(u, v) = \text{id} + \sum_{i,j=1}^N \frac{E_{ij} \otimes E_{ji}}{u - v} \in \text{End}(\mathbb{C}^N)^{\otimes 2}(u, v).$$

For any associative unital algebra  $X$  denote by  $\iota_s$  its embedding into a finite tensor product  $X^{\otimes n}$  as the  $s$ -th tensor factor:

$$\iota_s(x) = 1^{\otimes(s-1)} \otimes x \otimes 1^{\otimes(n-s)}, \quad x \in X; \quad s = 1, \dots, n.$$

Introduce the formal power series with the coefficients in  $\text{End}(\mathbb{C}^N)^{\otimes 2} \otimes Y(\mathfrak{gl}_N)$

$$T_1(u) = \iota_1 \otimes \text{id}(T(u)) \quad \text{and} \quad T_2(v) = \iota_2 \otimes \text{id}(T(v)).$$

Then the relations (1.2) divided by  $u - v$  can be rewritten as the single equality

$$(1.4) \quad R(u, v) \otimes 1 \cdot T_1(u) T_2(v) = T_2(v) T_1(u) \cdot R(u, v) \otimes 1.$$

Relations (1.2) imply that for any formal series  $f(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$  the assignment of generating series  $T_{ij}(u) \mapsto f(u) \cdot T_{ij}(u)$  defines an automorphism of the algebra  $Y(\mathfrak{gl}_N)$ . We will denote this automorphism by  $\omega_f$ .

The element  $T(u)$  of the algebra  $\text{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N)[[u^{-1}]]$  is invertible, let us denote

$$T(u)^{-1} = \tilde{T}(u) = \sum_{i,j=1}^N E_{ij} \otimes \tilde{T}_{ij}(u).$$

Then the relations (1.4) along with the equality  $R(u, v) R(-u, -v) = 1 - (u - v)^{-2}$  imply that the assignment  $T_{ij}(u) \mapsto \tilde{T}_{ij}(-u)$  determines an automorphism of the algebra  $Y(\mathfrak{gl}_N)$ . We will denote by this automorphism  $\sigma_N$ , it is clearly involutive.

Now consider the elements  $E_{ij}$  as generators of the Lie algebra  $\mathfrak{gl}_N$ . The algebra  $Y(\mathfrak{gl}_N)$  contains the enveloping algebra  $U(\mathfrak{gl}_N)$  as a subalgebra: due to (1.2) the assignment  $E_{ji} \mapsto T_{ij}^{(1)}$  defines the embedding. Moreover, there is a homomorphism

$$(1.5) \quad \pi_N : Y(\mathfrak{gl}_N) \rightarrow U(\mathfrak{gl}_N) : T_{ij}(u) \mapsto \delta_{ij} + E_{ji} u^{-1}.$$

Note that this homomorphism is identical on the subalgebra  $U(\mathfrak{gl}_N)$  by definition. We will fix the Borel subalgebra in  $\mathfrak{gl}_N$  generated by the elements  $E_{ij}$  with  $i \leq j$ . We will also fix the basis  $E_{11}, \dots, E_{NN}$  in the corresponding Cartan subalgebra.

For any non-negative integer  $M$  we will fix the standard embedding of Lie algebras  $\mathfrak{gl}_M \rightarrow \mathfrak{gl}_{M+N} : E_{ij} \mapsto E_{ij}$ . By (1.2) there is an embedding of algebras

$$\varphi : Y(\mathfrak{gl}_N) \rightarrow Y(\mathfrak{gl}_{M+N}) : T_{ij}(u) \mapsto T_{M+i, M+j}(u).$$

Consider the embedding of the same algebras  $\psi = \sigma_{M+N} \varphi \sigma_N$ . The following observation first appeared in [C2]; for its proof see also [O] and [NT, Section 1].

**Proposition 1.1.** *Image of the homomorphism  $\pi_{M+N} \circ \psi : Y(\mathfrak{gl}_N) \rightarrow U(\mathfrak{gl}_{M+N})$  commutes with the subalgebra  $U(\mathfrak{gl}_M)$  in  $U(\mathfrak{gl}_{M+N})$ .*

This proposition allows us to define a family of  $Y(\mathfrak{gl}_N)$ -modules which we will call *elementary*. For any pair of non-increasing sequences of integers

$$\lambda = (\lambda_1, \dots, \lambda_M, \lambda_{M+1}, \dots, \lambda_{M+N}) \quad \text{and} \quad \mu = (\mu_1, \dots, \mu_M)$$

denote by  $V_{\lambda, \mu}$  the subspace in the irreducible  $\mathfrak{gl}_{M+N}$ -module of highest weight  $\lambda$  formed by all singular vectors with respect to  $\mathfrak{gl}_M$  of weight  $\mu$ . This subspace is preserved by the action of the image of  $\pi_{M+N} \circ \psi$ . Thus  $V_{\lambda, \mu}$  becomes a module over the algebra  $Y(\mathfrak{gl}_N)$ . Relations (1.2) imply that for any  $h \in \mathbb{C}$  the assignment  $T_{ij}(u) \mapsto T_{ij}(u+h)$  determines an automorphism of the algebra  $Y(\mathfrak{gl}_N)$ ; here the series in  $(u+h)^{-1}$  should be re-expanded in  $u^{-1}$ . We will denote by  $V_{\lambda, \mu}(h)$  the  $Y(\mathfrak{gl}_N)$ -module obtained from  $V_{\lambda, \mu}$  by pulling back through this automorphism.

The study of the elementary modules  $V_{\lambda, \mu}(h)$  has been commenced in [C2] and continued in [NT]. Let us recall some of these results. There is a distinguished basis in the vector space  $V_{\lambda, \mu}$ . It constitutes a part of the *Gelfand-Zetlin basis* in the irreducible  $\mathfrak{gl}_{M+N}$ -module  $V_\lambda$  of the highest weight  $\lambda$ , corresponding to the chain of Lie subalgebras  $\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \dots \subset \mathfrak{gl}_{M+N}$ . The elements of the latter basis are labelled [GZ] by the arrays with integral entries

$$\Lambda = (\lambda_{mi} \mid m = 1, \dots, M+N; i = 1, \dots, m)$$

where  $\lambda_{M+N, i} = \lambda_i$  and  $\lambda_{mi} \geq \lambda_{m-1, i} \geq \lambda_{m, i+1}$  for all possible  $m$  and  $i$ . These arrays are called *Gelfand-Zetlin schemes* of type  $\lambda$ . For each scheme  $\Lambda$  there is a unique one-dimensional subspace  $V_\Lambda \subset V_\lambda$  which for every  $m = 1, \dots, M+N-1$  is contained in an irreducible  $\mathfrak{gl}_m$ -module of highest weight  $(\lambda_{m1}, \lambda_{m2}, \dots, \lambda_{mm})$ . By choosing a non-zero vector  $\xi_\Lambda$  in each subspace  $V_\Lambda$  one obtains a basis in  $V_\lambda$ .

The distinguished basis in  $V_{\lambda, \mu}$  is formed by the vectors  $\xi_\Lambda$  where the scheme  $\Lambda$  satisfies the condition  $\lambda_{mi} = \mu_i$  for every  $m = 1, \dots, M$ . We will denote by  $\mathcal{S}_{\lambda, \mu}$  the set of all such schemes. To describe the action of the Yangian  $Y(\mathfrak{gl}_N)$  on the vectors of this basis in  $V_{\lambda, \mu}(h)$  explicitly, it is convenient to use a set of generators different from  $T_{ij}^{(s)}$ . These alternative generators of the algebra  $Y(\mathfrak{gl}_N)$  are called the *Drinfeld generators* and can be defined as follows. Let  $\mathbf{i} = (i_1, \dots, i_k)$  and  $\mathbf{j} = (j_1, \dots, j_k)$  be any two sequences of integers such that

$$(1.6) \quad 1 \leq i_1 < \dots < i_k \leq N \quad \text{and} \quad 1 \leq j_1 < \dots < j_k \leq N.$$

Consider the alternating sum over all elements  $g$  of the symmetric group  $S_k$

$$(1.7) \quad \begin{aligned} Q_{\mathbf{i}\mathbf{j}}(u) &= \sum_g T_{i_1 j_{g(1)}}(u) T_{i_2 j_{g(2)}}(u-1) \dots T_{i_k j_{g(k)}}(u-k+1) \cdot \text{sgn } g \\ &= \sum_g T_{i_{g(k)} j_k}(u-k+1) \dots T_{i_{g(2)} j_2}(u-1) T_{i_{g(1)} j_1}(u) \cdot \text{sgn } g \end{aligned}$$

where the series in  $(u-1)^{-1}, \dots, (u-k+1)^{-1}$  should be re-expanded in  $u^{-1}$ . For the proof of the second equality here see [MNO, Section 2]. For each  $k = 1, \dots, N$  denote  $A_k(u) = Q_{\mathbf{i}\mathbf{i}}(u)$  where  $\mathbf{i} = (1, \dots, k)$ . Set  $A_0(u) = 1$ . The series  $A_N(u)$  is called the *quantum determinant* for the Yangian  $Y(\mathfrak{gl}_N)$ . The next proposition is well known, see [MNO, Section 2] for its detailed proof.

**Proposition 1.2.** *The coefficients at  $u^{-1}, u^{-2}, \dots$  of the series  $A_N(u)$  are free generators for the centre of the algebra  $Y(\mathfrak{gl}_N)$ .*

This proposition implies that all the coefficients of the series  $A_1(u), \dots, A_N(u)$  pairwise commute. Further, for each  $k = 1, \dots, N-1$  denote

$$(1.8) \quad B_k(u) = Q_{\mathbf{i}\mathbf{j}}(u), \quad C_k(u) = Q_{\mathbf{j}\mathbf{i}}(u), \quad D_k(u) = Q_{\mathbf{j}\mathbf{j}}(u)$$

where  $\mathbf{i} = (1, \dots, k)$  and  $\mathbf{j} = (1, \dots, k-1, k+1)$ . The coefficients of the series  $B_1(u), C_1(u), \dots, B_{N-1}(u), C_{N-1}(u)$  along with those of  $A_1(u), \dots, A_N(u)$  also generate [D2, Example] the algebra  $Y(\mathfrak{gl}_N)$ . It is the action of these generators of  $Y(\mathfrak{gl}_N)$  in  $V_{\lambda, \mu}(h)$  that can be calculated explicitly. For any  $k = 1, \dots, N$  denote

$$(1.9) \quad \rho_k(u) = \prod_{i=1}^M \frac{u + h + \mu_i - i - k + 1}{u + h - i - k + 1} \cdot \prod_{i=1}^{M+k} (u + h - i + 1).$$

Regard  $\rho_k(u)$  as a formal Laurent series in  $u^{-1}$ . From now on we will assume that the space  $V_{\lambda, \mu}$  is non-zero so that  $S_{\lambda, \mu} \neq \emptyset$ . Take any scheme  $\Lambda$  from  $S_{\lambda, \mu}$ .

**Theorem 1.3.** *For  $k = 1, \dots, N$  we have equality of formal Laurent series in  $u^{-1}$*

$$(1.10) \quad \rho_k(u) A_k(u) \cdot \xi_\Lambda = \xi_\Lambda \cdot \prod_{i=1}^{M+k} (u + h + \lambda_{M+k, i} - i + 1).$$

For  $k = 1, \dots, N-1$  formal Laurent series  $\rho_k(u) B_k(u) \cdot \xi_\Lambda$  and  $\rho_k(u) C_k(u) \cdot \xi_\Lambda$  in  $u^{-1}$  are actually polynomials in  $u$  and their degrees are less than  $M+k$ .

This theorem is contained in [NT, Section 2]. Let us now consider the zeroes

$$(1.11) \quad \nu_{ki} = i - h - \lambda_{M+k, i} - 1; \quad i = 1, \dots, M+k$$

of the polynomial in  $u$  at the right hand side of (1.10). Note that all these  $M+k$  zeroes are pairwise distinct since  $\lambda_{M+k, 1} \geq \dots \geq \lambda_{M+k, M+k}$  for any Gelfand-Zetlin scheme  $\Lambda$ . Therefore the polynomials  $\rho_k(u) B_k(u) \cdot \xi_\Lambda$  and  $\rho_k(u) C_k(u) \cdot \xi_\Lambda$  can be determined by their values at the points (1.11). To write down these values one has to make a choice of the vector  $\xi_\Lambda \in V_\Lambda$  for every  $\Lambda \in S_{\lambda, \mu}$ . Both these tasks have been performed in [NT, Section 3]. The next twin theorems are weaker than those results of [NT] but will suffice for our present purposes. Let the indices  $k \in \{1, \dots, N-1\}$  and  $i \in \{1, \dots, M+k\}$  be fixed. Denote by  $\Lambda^-$  and  $\Lambda^+$  the arrays obtained from  $\Lambda$  by decreasing and increasing the  $(M+k, i)$ -entry by 1.

**Theorem 1.4.** *If  $\Lambda^- \in S_{\lambda, \mu}$  then the image  $\rho_k(u) B_k(u) \cdot V_\Lambda$  at  $u = \nu_{ki}$  is  $V_{\Lambda^-}$ . If otherwise  $\Lambda^- \notin S_{\lambda, \mu}$  then this image at  $u = \nu_{ki}$  is zero.*

**Theorem 1.5.** *If  $\Lambda^+ \in S_{\lambda, \mu}$  then the image  $\rho_k(u) C_k(u) \cdot V_\Lambda$  at  $u = \nu_{ki}$  is  $V_{\Lambda^+}$ . If otherwise  $\Lambda^+ \notin S_{\lambda, \mu}$  then this image at  $u = \nu_{ki}$  is zero.*

The above three theorems imply that the elementary  $Y(\mathfrak{gl}_N)$ -module  $V_{\lambda, \mu}(h)$  is irreducible. Let us now point out the place of elementary modules in the family of all irreducible finite-dimensional  $Y(\mathfrak{gl}_N)$ -modules. Let  $V$  be any module from the latter family. A non-zero vector  $\zeta$  in any  $Y(\mathfrak{gl}_N)$ -module is called *singular* if it is annihilated by all the coefficients of the series  $C_1(u), \dots, C_{N-1}(u)$ . The

vector  $\zeta \in V$  is then unique up to a scalar multiplier and is an eigenvector for the coefficients of the series  $A_1(u), \dots, A_N(u)$ ; see [D2, Theorem 2]. Moreover, then

$$(1.12) \quad \frac{A_{k+1}(u)A_{k-1}(u-1)}{A_k(u)A_k(u-1)} \cdot \zeta = \frac{P_k(u-1)}{P_k(u)} \cdot \zeta; \quad k = 1, \dots, N-1$$

for certain monic polynomials  $P_1(u), \dots, P_{N-1}(u)$  with coefficients in  $\mathbb{C}$ . These  $N-1$  polynomials are called the *Drinfeld polynomials* of the module  $V$ . Every collection of  $N-1$  monic polynomials arises in this way. The modules with the same Drinfeld polynomials may differ only by an automorphism of the algebra  $Y(\mathfrak{gl}_N)$  of the form  $\omega_f$ . Now consider the scheme  $\Lambda^\circ \in \mathcal{S}_{\lambda, \mu}$  with the entries

$$\lambda_{mi}^\circ = \begin{cases} \mu_i & \text{if } m \leq M, \\ \min(\lambda_i, \mu_{i-m+M}) & \text{if } m > M \text{ and } i > m - M, \\ \lambda_i & \text{if } m > M \text{ and } i \leq m - M. \end{cases}$$

Observe that  $\lambda_{mi}^\circ \geq \lambda_{mi}$  for every scheme  $\Lambda \in \mathcal{S}_{\lambda, \mu}$ . Therefore by Theorem 1.5 the vector  $\xi_{\Lambda^\circ} \in V_{\lambda, \mu}(h)$  is singular. Theorem 1.3 then allows to find the Drinfeld polynomials of the module  $V_{\lambda, \mu}(h)$ . We again refer to [NT, Section 2] for details.

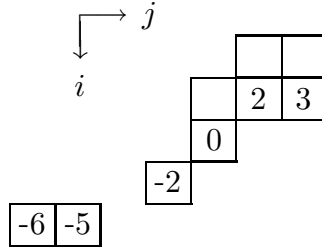
Here we will only formulate the answer. The assumption  $S_{\lambda, \mu} \neq \emptyset$  implies that

$$(1.13) \quad \lambda_1 \geq \mu_1, \dots, \lambda_M \geq \mu_M \quad \text{and} \quad \mu_M \geq \lambda_{M+N}.$$

Consider the *skew Young diagram*  $\lambda/\mu$ . This is the set of pairs

$$\{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq M+N, \lambda_i \geq j > \mu_i\}$$

where for any  $i > M$  we write  $\mu_i = \lambda_{M+N}$ . Employ usual graphic representation of a diagram: the point  $(i, j) \in \mathbb{Z}^2$  is represented by the unit box on the plane  $\mathbb{R}^2$  with the centre  $(i, j)$ ; the coordinates  $i$  and  $j$  on  $\mathbb{R}^2$  increasing from top to bottom and from left to right respectively. The *content* of the box corresponding to  $(i, j)$  is the difference  $c = j - i$ . Here is the diagram corresponding to  $\lambda = (5, 5, 3, 2, 0, -2)$  and  $\mu = (3, 2, 2, 1)$ ; we indicate the content of the bottom box for every column.



The condition  $S_{\lambda, \mu} \neq \emptyset$  is then equivalent to (1.13) along with the requirement that any column of the skew diagram  $\lambda/\mu$  has at most  $N$  boxes; see [M, Section I.5].

**Proposition 1.6.** *The Drinfeld polynomials of the  $Y(\mathfrak{gl}_N)$ -module  $V_{\lambda, \mu}(h)$  are*

$$P_k(u) = \prod_c (u + h + c); \quad k = 1, \dots, N-1$$

where the product is taken over the contents of the bottom boxes in the columns of height  $k$  in the skew Young diagram  $\lambda/\mu$ .

Let us equip the algebra  $Y(\mathfrak{gl}_N)$  with the  $\mathbb{Z}$ -grading  $\deg$  determined by

$$(1.14) \quad \deg T^{(s)} = \begin{pmatrix} s & 0 & 0 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & 0 & 0 \\ 0 & 0 & 0 & s & 0 & 0 \\ 0 & 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 0 & 0 & s \end{pmatrix}$$

see defining relations (1.2). We will extend this grading to the algebra  $Y(\mathfrak{gl}_N)[[u^{-1}]]$  by assuming that  $\deg u^{-1} = 0$ . Then  $\deg A_k(u) = 0$  while by the definition (1.8)

$$\deg B_k(u) = -1, \quad \deg C_k(u) = 1.$$

The vector space  $V_{\lambda,\mu}$  has a natural  $\mathbb{Z}$ -grading by eigenvalues of the action of

$$(M+N)E_{11} + (M+N-1)E_{22} + \dots + E_{M+N,M+N} \in \mathfrak{gl}_{M+N}.$$

Any subspace  $V_\Lambda$  in  $V_{\lambda,\mu}$  is homogeneous and its degree equals the sum of  $\lambda_{mi}$  over all indices  $m = 1, \dots, M+N$  and  $i = 1, \dots, m$ . The above three theorems show that the action of the algebra  $Y(\mathfrak{gl}_N)$  in the module  $V_{\lambda,\mu}(h)$  is graded. The singular vector  $\xi_{\Lambda^\circ} \in V_{\lambda,\mu}(h)$  then has the maximal degree. Hence it is an eigenvector for all the coefficients of the series  $T_{11}(u), \dots, T_{NN}(u)$ . Let  $R_1(u), \dots, R_N(u)$  be the corresponding eigenvalues. Let  $J$  be the left ideal in  $Y(\mathfrak{gl}_N)[[u^{-1}]]$  generated by the elements of positive  $\mathbb{Z}$ -degrees. By definition  $A_k(u)$  equals

$$T_{11}(u)T_{22}(u-1)\dots T_{kk}(u-k+1)$$

plus certain elements from the ideal  $J$ . Therefore from (1.12) we get the equalities

$$(1.15) \quad \frac{P_k(u-1)}{P_k(u)} = \frac{R_{k+1}(u-k)}{R_k(u-k)}; \quad k = 1, \dots, N-1.$$

Due to the defining relations (1.2) the assignment  $T_{ij}(u) \mapsto T_{ji}(u)$  determines an anti-automorphism of the algebra  $Y(\mathfrak{gl}_N)$ . Denote by  $\theta$  this anti-automorphism, it is obviously involutive. Now for any finite-dimensional  $Y(\mathfrak{gl}_N)$ -module  $W$  define its *dual* module  $W^*$  as the vector space dual to  $W$  where  $\langle y \cdot \xi^*, \xi \rangle = \langle \xi^*, \theta(y) \cdot \xi \rangle$  for any  $\xi \in W$ ,  $\xi^* \in W^*$  and  $y \in Y(\mathfrak{gl}_N)$ . We will need the following simple fact.

**Proposition 1.7.** *Any elementary  $Y(\mathfrak{gl}_N)$ -module  $V_{\lambda,\mu}(h)$  is self-dual.*

*Proof.* Let  $\zeta \in V_{\lambda,\mu}(h)$  be a singular vector. It spans the subspace of the maximal  $\mathbb{Z}$ -degree, so there is an element  $\zeta^* \in V_{\lambda,\mu}(h)^*$  with  $\langle \zeta^*, \zeta \rangle = 1$  and  $\langle \zeta^*, \xi \rangle = 0$  for any vector  $\xi \in V_{\lambda,\mu}(h)$  with non-maximal degree. But thanks to (1.7) we have

$$\theta(A_k(u)) = A_k(u), \quad \theta(B_k(u)) = C_k(u), \quad \theta(C_k(u)) = B_k(u)$$

for all possible indices  $k$ . Therefore the vector  $\zeta^*$  is singular in  $V_{\lambda,\mu}(h)^*$  and the eigenvalues of  $A_1(u), \dots, A_N(u)$  on this vector are the same as on the vector  $\zeta$  respectively. So the  $Y(\mathfrak{gl}_N)$ -modules  $V_{\lambda,\mu}(h)^*$  and  $V_{\lambda,\mu}(h)$  are equivalent  $\square$

There is a natural Hopf algebra structure on the  $Y(\mathfrak{gl}_N)$ . The antipode is defined by the assignment of generating series  $T_{ij}(u) \mapsto \tilde{T}_{ij}(u)$  while the comultiplication  $Y(\mathfrak{gl}_N) \rightarrow Y(\mathfrak{gl}_N)^{\otimes 2}$  is defined by the assignment

$$(1.16) \quad T_{ij}(u) \mapsto \sum_{k=1}^N T_{ik}(u) \otimes T_{kj}(u).$$

Here and in what follows we take tensor products of the elements of the algebra  $Y(\mathfrak{gl}_N)[[u^{-1}]]$  over its subalgebra  $\mathbb{C}[[u^{-1}]]$ . We will also use the comultiplication

$\Delta'$  on the algebra  $Y(\mathfrak{gl}_N)$  obtained by composing  $\Delta$  with the transposition of tensor factors in  $Y(\mathfrak{gl}_N)^{\otimes 2}$ . Observe that by the definition (1.16) we have

$$(1.17) \quad \Delta \circ \theta = (\theta \otimes \theta) \circ \Delta'.$$

We will consider the images of the Drinfeld generators for the algebra  $Y(\mathfrak{gl}_N)$  with respect to the  $n$ -fold comultiplication

$$(1.18) \quad \Delta^{(n)} : Y(\mathfrak{gl}_N) \rightarrow Y(\mathfrak{gl}_N)^{\otimes n}.$$

For this purpose we will employ the following easy result from [NT, Section 1]. Let  $\mathbf{i}$  and  $\mathbf{j}$  be any two sequences of indices satisfying the condition (1.6).

**Proposition 1.8.** *We have the equality*

$$\Delta^{(n)}(Q_{\mathbf{ij}}(u)) = \sum_{\mathbf{k}^{(1)}, \mathbf{k}^{(2)}, \dots, \mathbf{k}^{(n-1)}} Q_{\mathbf{i}\mathbf{k}^{(1)}}(u) \otimes Q_{\mathbf{k}^{(1)}\mathbf{k}^{(2)}}(u) \otimes \dots \otimes Q_{\mathbf{k}^{(n-1)}\mathbf{j}}(u)$$

where  $\mathbf{k}^{(1)}, \mathbf{k}^{(2)}, \dots, \mathbf{k}^{(n-1)}$  are increasing sequences of integers  $1, \dots, N$  of length  $k$ .

With the  $\mathbb{Z}$ -grading  $\deg$  on  $Y(\mathfrak{gl}_N)$ , the algebras  $Y(\mathfrak{gl}_N)^{\otimes n}$  and  $Y(\mathfrak{gl}_N)^{\otimes n}[[u^{-1}]]$  acquire grading by the group  $\mathbb{Z}^n$ . In the next section we will give an alternative realization of the elementary  $Y(\mathfrak{gl}_N)$ -module  $V_{\lambda, \mu}(h)$  using comultiplication (1.18).

## 2. Intertwining operators

In this section we study intertwining operators between the tensor products of two  $Y(\mathfrak{gl}_N)$ -modules of the form  $V_{\lambda, \mu}(h)$  defined via the comultiplications  $\Delta$  and  $\Delta'$  on the algebra  $Y(\mathfrak{gl}_N)$ . We use the explicit realization [C2] of the module  $V_{\lambda, \mu}(h)$ .

Consider first the *vector*  $Y(\mathfrak{gl}_N)$ -module  $V(h)$ . This is the elementary module  $V_{\lambda, \mu}(h)$  with  $M = 0$  and  $\lambda = (1, 0, \dots, 0)$ . So the space of this module is  $\mathbb{C}^N$  and the action of the algebra  $Y(\mathfrak{gl}_N)$  is defined by  $T_{ij}^{(s)} \mapsto E_{ji} \cdot h^{s-1}$  for any  $s \geq 1$ . Note that due to (1.3) this action can be then determined by the single assignment

$$(2.1) \quad \text{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N)[[u^{-1}]] \rightarrow \text{End}(\mathbb{C}^N)^{\otimes 2}[[u^{-1}]] : T(u) \mapsto R(u, -h).$$

Now take any non-zero elementary  $Y(\mathfrak{gl}_N)$ -module  $V_{\lambda, \mu}(h)$ . Let  $n$  be the total number of boxes in the skew Young diagram  $\lambda/\mu$ . Consider the *row tableau* of shape  $\lambda/\mu$  obtained by filling the boxes of  $\lambda$  with  $1, \dots, n$  in the natural order, that is by rows downwards, from the left to right in every row. Denote by  $\Omega$  this tableau. Here is the row tableau corresponding to  $\lambda = (5, 5, 3, 2, 0, -2)$  and  $\mu = (3, 2, 2, 1)$ :

			1	2
		3	4	5
		6		
	7			
8	9			

Denote by  $S_{\lambda/\mu}$  and  $T_{\lambda/\mu}$  the subgroups in the symmetric group  $S_n$  preserving the sets of numbers appearing respectively in every row and in every column of  $\Omega$ .

The symmetric group  $S_n$  acts in the space  $(\mathbb{C}^N)^{\otimes n}$  by permutations of the tensor factors. Let  $P_{\lambda/\mu}$  and  $Q_{\lambda/\mu}$  be the elements of  $\text{End}(\mathbb{C}^N)^{\otimes n}$  corresponding to the sums

$$\sum_{g \in S_{\lambda/\mu}} g \quad \text{and} \quad \sum_{g \in T_{\lambda/\mu}} g \cdot \text{sgn } g$$

in  $\mathbb{C} \cdot S_n$ . The product  $Y_{\lambda/\mu} = P_{\lambda/\mu} Q_{\lambda/\mu}$  is the *Young symmetrizer* corresponding to the tableau  $\Omega$ . Let  $c_1, \dots, c_n$  be the contents of the boxes of  $\lambda/\mu$  occupied respectively by the numbers  $1, \dots, n$  in the row tableau  $\Omega$ . Then consider the  $Y(\mathfrak{gl}_N)$ -module obtained from the tensor product  $V(c_1 + h) \otimes \dots \otimes V(c_n + h)$  via the comultiplication (1.18).

**Proposition 2.1.** *Action of the algebra  $Y(\mathfrak{gl}_N)$  in  $V(c_1 + h) \otimes \dots \otimes V(c_n + h)$  preserves the image of the operator  $Y_{\lambda/\mu}$ . The module  $V_{\lambda,\mu}(h)$  can be obtained from this image by pulling back through an automorphism of the form  $\omega_f$ .*

*Proof.* Consider the  $Y(\mathfrak{gl}_N)$ -module  $V(c_1 + h) \otimes \dots \otimes V(c_n + h)$ . By (1.16) and (2.1) the action of  $Y(\mathfrak{gl}_N)$  in this module can be determined by the assignment

$$(2.2) \quad \begin{aligned} \text{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N)[[u^{-1}]] &\rightarrow \text{End}(\mathbb{C}^N)^{\otimes(n+1)}[[u^{-1}]] : \\ T(u) &\mapsto R_{12}(u, -c_1 - h) \dots R_{1,n+1}(u, -c_n - h). \end{aligned}$$

Here we use the standard notation: for any  $1 \leq p < q \leq m$  and  $X \in \text{End}(\mathbb{C}^N)^{\otimes 2}$  we write

$$X_{pq} = \iota_p \otimes \iota_q (X) \in \text{End}(\mathbb{C}^N)^{\otimes m}.$$

Introduce the *inverse column tableau* of shape  $\lambda/\mu$ . This tableau is obtained by filling the boxes of  $\lambda/\mu$  with  $1, \dots, n$  by columns from the right to the left, upwards in every column. Denote it by  $\Omega^*$ . Here is the tableau  $\Omega^*$  corresponding to the same sequences  $\lambda = (5, 5, 3, 2, 0, -2)$  and  $\mu = (3, 2, 2, 1)$  as before:

			4	2
		6	3	1
		5		
	7			
9	8			

Let the symmetric group  $S_n$  act on the entries of the tableau  $\Omega^*$ . Consider the permutation  $g \in S_n$  such that  $g : \Omega^* \mapsto \Omega$ . Then [C1, Theorem 1] provides the equality in the algebra  $\text{End}(\mathbb{C}^N)^{\otimes(n+1)}[[u^{-1}]]$

$$(2.3) \quad \begin{aligned} R_{12}(u, -c_1 - h) \dots R_{1,n+1}(u, -c_n - h) \cdot (1 \otimes Y_{\lambda/\mu}) = \\ (1 \otimes Y_{\lambda/\mu}) \cdot R_{1,g(1)+1}(u, -c_{g(1)} - h) \dots R_{1,g(n)+1}(u, -c_{g(n)} - h). \end{aligned}$$

Along with (2.2) this equality yields the first statement of Proposition 2.1. We will denote by  $V_{\lambda/\mu}(h)$  the image of the operator  $Y_{\lambda/\mu}$  regarded as  $Y(\mathfrak{gl}_N)$ -module.

Further, the symmetric group  $S_n$  acts on the algebra  $Y(\mathfrak{gl}_N)^{\otimes n}$  by permutations of the tensor factors. Consider the  $Y(\mathfrak{gl}_N)$ -module  $U$  obtained from the product

$V(c_1 + h) \otimes \dots \otimes V(c_n + h)$  by composing the comultiplication  $\Delta^{(n)}$  with the above permutation  $g$ . The action of the algebra  $Y(\mathfrak{gl}_N)$  in  $U$  can be then described by

$$\begin{aligned} \text{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N)[[u^{-1}]] &\rightarrow \text{End}(\mathbb{C}^N)^{\otimes(n+1)}[[u^{-1}]] : \\ T(u) &\mapsto R_{1,g(1)+1}(u, -c_{g(1)}-h) \dots R_{1,g(n)+1}(u, -c_{g(n)}-h). \end{aligned}$$

Equality (2.3) also shows that this action preserves the kernel of the operator  $Y_{\lambda/\mu}$ . Moreover, by (2.3) the  $Y(\mathfrak{gl}_N)$ -module  $V_{\lambda/\mu}(h)$  is equivalent to the quotient of  $U$  by this kernel. We will show that this quotient is irreducible and has the same Drinfeld polynomials as the elementary  $Y(\mathfrak{gl}_N)$ -module  $V_{\lambda,\mu}(h)$ .

First consider the case when the skew Young diagram  $\lambda/\mu$  consists of one column only. By our assumption the length  $n$  of this column does not exceed  $N$ . Then we have  $c_p = c_1 - p + 1$  and  $g(p) = n - p + 1$  for each  $p = 1, \dots, n$  while  $Y_{\lambda/\mu}$  is the operator of antisymmetrization in  $(\mathbb{C}^N)^{\otimes n}$ . Further, then for

$$P = \sum_{i,j=1}^N E_{ij} \otimes E_{ji} \in \text{End}(\mathbb{C}^N)^{\otimes 2}$$

we have the equality of rational functions in  $u$  valued in the algebra  $\text{End}(\mathbb{C}^N)^{\otimes(n+1)}$

$$\begin{aligned} (1 \otimes Y_{\lambda/\mu}) \cdot R_{1,n+1}(u, -c_n-h) \dots R_{12}(u, -c_1-h) \\ = (1 \otimes Y_{\lambda/\mu}) \cdot \left( 1 + \frac{P_{12} + \dots + P_{1,n+1}}{u + h + c_1} \right); \end{aligned}$$

see for instance [N, Proposition 2.12]. This equality along with (1.5) shows that the quotient of  $U$  by  $\ker Y_{\lambda/\mu}$  is equivalent to the  $Y(\mathfrak{gl}_N)$ -module  $V_{\nu,\emptyset}(c_1 + h)$  where  $\nu$  is the  $n$ -th fundamental  $\mathfrak{gl}_N$ -weight  $(1, \dots, 1, 0, \dots, 0)$ . Let  $e_1, \dots, e_N$  be the standard basis in  $\mathbb{C}^N$ . Due to Proposition 1.6 every Drinfeld polynomial of the  $Y(\mathfrak{gl}_N)$ -module  $V_{\nu,\emptyset}(c_1 + h)$  is 1, except for the  $n$ -th when  $n < N$ . Then the  $n$ -th polynomial is

$$u + c_1 + h + 1 - n = u + h + c_n.$$

The Drinfeld polynomials of the elementary module  $V_{\lambda,\mu}(h)$  are the same. So we obtain the required statement in the one-column case. Observe that in this case the image of the vector  $e_1 \otimes \dots \otimes e_n \in U$  in the quotient by  $\ker Y_{\lambda/\mu}$  is singular.

Now consider the case of arbitrary  $\lambda$  and  $\mu$ . Let us denote by  $\mathcal{S}_{\lambda/\mu}$  the set of *semi-standard tableaux* of shape  $\lambda/\mu$  with entries  $1, \dots, N$ . These tableaux are the functions  $\kappa : \{1, \dots, n\} \rightarrow \{1, \dots, N\}$  such that  $\kappa(p) < \kappa(q)$  or  $\kappa(p) \leq \kappa(q)$  if the numbers  $p < q$  appear respectively in the same column or the same row of  $\Omega$ . Every tableau  $\kappa \in \mathcal{S}_{\lambda/\mu}$  determines a vector  $e_{\kappa(1)} \otimes \dots \otimes e_{\kappa(n)} \in (\mathbb{C}^N)^{\otimes n}$ . The images of all these vectors in the quotient of  $(\mathbb{C}^N)^{\otimes n}$  by  $\ker Y_{\lambda/\mu}$  form a basis in this quotient; see for instance [JK, Section 7.2]. Moreover, the sets  $\mathcal{S}_{\lambda/\mu}$  and  $\mathcal{S}_{\lambda,\mu}$  are of the same cardinality. Therefore it suffices to point out a singular vector  $\zeta$  in the quotient by  $\ker Y_{\lambda/\mu}$  of the  $Y(\mathfrak{gl}_N)$ -module  $U$  such that the equalities (1.12) hold for the Drinfeld polynomials  $P_1(u), \dots, P_{N-1}(u)$  of the module  $V_{\lambda,\mu}(h)$ . In particular, we will then obtain that this quotient is irreducible.

Let  $m$  be the number of columns in the skew diagram  $\lambda/\mu$ . By our assumption the length of any column does not exceed  $N$ . Let  $\alpha^0(x) \in \{1, \dots, N\}$  be the depth

of the box of the tableau  $\Omega$  with the number  $p$  in its column. Then  $\kappa^\circ \in \mathcal{S}_{\lambda/\mu}$ . Arguments already used in the one-column case show that the action of the algebra  $Y(\mathfrak{gl}_N)$  in  $U$  preserves the kernel of the operator  $Q_{\lambda/\mu}$ . The image  $\eta$  of the vector  $e_{\kappa^\circ(1)} \otimes \dots \otimes e_{\kappa^\circ(n)} \in U$  in the quotient by  $\ker Q_{\lambda/\mu}$  is singular. This follows from Proposition 1.8 applied to the number  $m$  instead of  $n$ , see also (1.14). Moreover, the same proposition shows that for any  $k = 1, \dots, N$  the coproduct  $\Delta^{(m)}(A_k(u))$  equals  $A_k(u)^{\otimes m}$  plus terms with degrees in  $\mathbb{Z}^m$  containing at least one positive component. Using the results of the one-column case, we then obtain the equalities

$$(2.4) \quad \frac{A_{k+1}(u) A_{k-1}(u-1)}{A_k(u) A_k(u-1)} \cdot \eta = \eta \cdot \prod_c \frac{u+h+c-1}{u+h+c}; \quad k = 1, \dots, N-1$$

where the product is taken over the contents of the bottom boxes in the columns of height  $k$  in the skew Young diagram  $\lambda/\mu$ . Now let  $\zeta$  be the image of the vector  $e_{\kappa^\circ(1)} \otimes \dots \otimes e_{\kappa^\circ(n)} \in U$  in the quotient by  $\ker Y_{\lambda/\mu}$ . Since  $\ker Q_{\lambda/\mu} \subset \ker Y_{\lambda/\mu}$ , the vector  $\zeta$  is singular in this quotient  $Y(\mathfrak{gl}_N)$ -module. Moreover, the equalities (2.4) along with Proposition 1.6 imply (1.12) for this quotient and for the Drinfeld polynomials  $P_1(u), \dots, P_{N-1}(u)$  of the elementary module  $V_{\lambda,\mu}(h)$   $\square$

Now fix any two skew Young diagrams  $\alpha$  and  $\beta$ . Let  $m$  and  $n$  be the numbers of boxes in  $\alpha$  and  $\beta$  respectively. Let  $z$  be a complex parameter as well as  $h$ . Consider the irreducible  $Y(\mathfrak{gl}_N)$ -modules  $V_\alpha(h)$  and  $V_\beta(z)$ . Let  $a_1, \dots, a_m$  and  $b_1, \dots, b_n$  be the contents of the boxes of  $\alpha$  and  $\beta$  occupied by the numbers  $1, \dots, m$  and  $1, \dots, n$  in the corresponding row tableaux. Then  $V_\alpha(h)$  and  $V_\beta(z)$  are the submodules in  $V(a_1+h) \otimes \dots \otimes V(a_m+h)$  and  $V(b_1+z) \otimes \dots \otimes V(b_n+z)$  defined as the images of Young symmetrizers  $Y_\alpha$  and  $Y_\beta$  in  $(\mathbb{C}^N)^{\otimes m}$  and  $(\mathbb{C}^N)^{\otimes n}$  respectively; see the proof of Proposition 2.1.

We assume that the  $Y(\mathfrak{gl}_N)$ -modules  $V_\alpha(h)$  and  $V_\beta(z)$  are both non-zero. So the length of any column in  $\alpha$  and  $\beta$  does not exceed  $N$ . Note that if  $m = n$  while  $a_k = b_k + c$  for each  $k = 1, \dots, m$  and the same integer  $c$  then  $V_\alpha(h) = V_\beta(h+c)$ .

Equip the set all of pairs  $(k, l)$  where  $k = 1, \dots, m$  and  $l = 1, \dots, n$  with the following ordering: the pair  $(i, j)$  precedes  $(k, l)$  if  $i > k$ , or if  $i = k$  but  $j < l$ . Using this ordering introduce the rational function in  $h, z$  valued in  $\text{End}(\mathbb{C}^N)^{\otimes(m+n)}$

$$(2.5) \quad \prod_{(k,l)}^{\rightarrow} R_{k,m+l}(-a_k-h, -b_l-z) \cdot Y_\alpha \otimes Y_\beta$$

where  $Y_\alpha$  acts non-trivially only on the first  $m$  tensor factors in  $(\mathbb{C}^N)^{\otimes(m+n)}$  while  $Y_\beta$  acts only on the last  $n$  tensor factors. By the definition (1.3) this function depends only on the difference  $h - z$ . Due to (2.3) the expression (2.5) is divisible by  $1 \otimes Y_\beta$  also on the left. Furthermore, since  $R(u, v)R(v, u) = 1 - (u - v)^{-2}$  the equality (2.3) in the algebra  $\text{End}(\mathbb{C}^N)^{\otimes(n+1)}[[u^{-1}]]$  can be rewritten as

$$R_{n,n+1}(-c_n-h, u) \dots R_{1,n+1}(-c_1-h, u) \cdot (Y_{\lambda/\mu} \otimes 1) = (Y_{\lambda/\mu} \otimes 1) \cdot R_{g(n),n+1}(-c_{g(n)}-h, u) \dots R_{g(1),n+1}(-c_{g(1)}-h, u).$$

Hence the expression (2.5) is divisible by  $Y_\alpha \otimes 1$  on the left. Thus (2.5) determines a rational function in  $h - z$  valued in  $\text{End}(\text{im } Y_\alpha \otimes \text{im } Y_\beta)$ . Denote by  $R_{\alpha\beta}(h)$  the first non-zero coefficient of the Laurent series in  $z$  of this function at  $z = 0$ .

Now let  $W$  and  $W'$  be the  $Y(\mathfrak{gl}_N)$ -modules obtained from the tensor product  $V_\alpha(h) \otimes V_\beta(0)$  via the comultiplications  $\Delta$  and  $\Delta'$  respectively. Then the element  $R_{\alpha\beta}(h) \in \text{End}(\text{im } Y_\alpha \otimes \text{im } Y_\beta)$  admits the following interpretation

**Proposition 2.2.** *The coefficient  $R_{\alpha\beta}(h)$  is an intertwining operator  $W' \rightarrow W$ .*

*Proof.* Let us denote by  $U$  and  $U'$  the  $Y(\mathfrak{gl}_N)$ -modules obtained respectively via the comultiplications  $\Delta$  and  $\Delta'$  from the tensor product of the  $Y(\mathfrak{gl}_N)$ -modules  $V(a_1 + h) \otimes \dots \otimes V(a_m + h)$  and  $V(b_1 + z) \otimes \dots \otimes V(b_n + z)$ . The action of the algebra  $Y(\mathfrak{gl}_N)$  in the module  $U$  can be determined by the assignment

$$\begin{aligned} \text{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N)[[u^{-1}]] &\rightarrow \text{End}(\mathbb{C}^N)^{\otimes(m+n+1)}[[u^{-1}]] : \\ T(u) &\mapsto R_{12}(u, -a_1 - h) \dots R_{1,m+1}(u, -a_m - h) \times \\ &\quad R_{1,m+2}(u, -b_1 - z) \dots R_{1,m+n+1}(u, -b_n - z); \end{aligned}$$

cf. (2.2). The action of  $Y(\mathfrak{gl}_N)$  in  $U'$  can be then determined by the assignment

$$\begin{aligned} \text{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N)[[u^{-1}]] &\rightarrow \text{End}(\mathbb{C}^N)^{\otimes(m+n+1)}[[u^{-1}]] : \\ T(u) &\mapsto R_{1,m+2}(u, -b_1 - z) \dots R_{1,m+n+1}(u, -b_n - z) \times \\ &\quad R_{12}(u, -a_1 - h) \dots R_{1,m+1}(u, -a_m - h). \end{aligned}$$

By applying repeatedly the *Yang-Baxter equation* in  $\text{End}(\mathbb{C}^N)^{\otimes 3}(u, v, w)$

$$R_{12}(u, v) R_{13}(u, w) R_{23}(v, w) = R_{23}(v, w) R_{13}(u, w) R_{12}(u, v)$$

we obtain the equality of rational functions in  $h, z$  valued in  $\text{End}(\mathbb{C}^N)^{\otimes(m+n+1)}$

$$\begin{aligned} &\prod_k^{\rightarrow} R_{1,k+1}(u, -a_k - h) \cdot \prod_l^{\rightarrow} R_{1,m+l+1}(u, -b_l - z) \times \\ &\quad \prod_{(i,j)}^{\rightarrow} R_{k+1,m+l+1}(-a_k - h, -b_l - z) \cdot 1 \otimes Y_\alpha \otimes Y_\beta = \\ &\quad \prod_{(k,l)}^{\rightarrow} R_{k+1,m+l+1}(-a_k - h, -b_l - z) \times \\ &\quad \prod_l^{\rightarrow} R_{1,m+l+1}(u, -b_l - z) \cdot \prod_k^{\rightarrow} R_{1,k+1}(u, -a_k - h) \cdot 1 \otimes Y_\alpha \otimes Y_\beta. \end{aligned}$$

Here the index  $k$  runs through the set  $\{1, \dots, m\}$  while  $l$  runs through  $\{1, \dots, n\}$ . Note that the expression in the last line of this equality is divisible by  $1 \otimes Y_\alpha \otimes Y_\beta$  also on the left. Let  $d$  be the degree of the first non-zero term in the Laurent series of the rational function (2.5) in  $z$  at  $z = 0$ . Dividing the above equality by  $z^d$  and then tending  $z \rightarrow 0$ , we obtain Proposition 2.2  $\square$

By the definition (2.5) the operator  $R_{\alpha\beta}(h)$  in  $\text{im } Y_\alpha \otimes \text{im } Y_\beta$  is invertible for every  $h \in \mathbb{C} \setminus \mathbb{Z}$ . Now consider the following special situation. Suppose that the diagram  $\beta$  is a usual Young diagram. Thus for certain integers  $\beta_1 \geq \dots \geq \beta_N \geq 0$  we have

Note that in this case  $b_1 = 0$ . Further, suppose that  $\alpha$  is a *reversed* Young diagram:

$$\alpha = \{ (i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq N, N - \alpha_{N-i+1} < j \leq N \}$$

for certain integers  $\alpha_1 \geq \dots \geq \alpha_N \geq 0$ . Note that then  $a_m = 0$ . For example, here for  $N = 3$  we show the reversed and the usual Young diagrams corresponding to  $(\alpha_1, \alpha_2, \alpha_3) = (\beta_1, \beta_2, \beta_3) = (4, 3, 1)$ . We have indicated the contents  $a_8 = b_1 = 0$ .



In this special situation one can describe explicitly the set of all points  $h \in \mathbb{Z}$  where the operator  $R_{\alpha\beta}(h)$  is non-invertible. It is the main result of this section, cf. [DO].

**Theorem 2.3.** *The operator  $R_{\alpha\beta}(h)$  is not invertible at  $h \in \mathbb{Z}$  if and only if*

$$\alpha_i + \beta_N, \alpha_{i+1} + \beta_{N-1}, \dots, \alpha_N + \beta_i < h + i \leq \alpha_1 + \beta_i, \alpha_2 + \beta_{i-1}, \dots, \alpha_i + \beta_1$$

for at least one index  $i \in \{1, \dots, N\}$ .

*Proof.* By applying [N, Proposition 2.12] to the usual Young diagram  $\beta$  and by employing the equality  $b_1 = 0$  we can bring the product (2.5) to the form

$$(2.6) \quad \prod_k^{\leftarrow} \left( 1 - \frac{P_{k,m+1} + \dots + P_{k,m+n}}{h + a_k - z} \right) \cdot Y_\alpha \otimes Y_\beta$$

where the factors corresponding to  $k = 1, \dots, m$  are arranged from right to left. Let us use the following well-known property of the element  $Y_\alpha \in \text{End}(\mathbb{C}^N)^{\otimes m}$ :

$$(2.7) \quad (P_{k,k+1} + \dots + P_{km}) \cdot Y_\alpha = -a_k Y_\alpha,$$

see [C1, Theorem 4] and [J, Section 4]. For each  $k = 1, \dots, m$  consider the element

$$X_k = P_{k,k+1} + \dots + P_{km} + P_{k,m+1} + \dots + P_{k,m+n}$$

of the algebra  $\text{End}(\mathbb{C}^N)^{\otimes(m+n)}$ . The sum  $P_{k,k+1} + \dots + P_{km}$  in  $\text{End}(\mathbb{C}^N)^{\otimes(m+n)}$  commutes with each of the elements  $X_1, \dots, X_{k-1}$ . Therefore by applying (2.7) consecutively to  $k = 1, \dots, m$  we can rewrite the product (2.6) as

$$(2.8) \quad \prod_k \left( 1 - \frac{X_k + a_k}{h + a_k - z} \right) \cdot Y_\alpha \otimes Y_\beta = \prod_k \frac{h - X_k - z}{h + a_k - z} \cdot Y_\alpha \otimes Y_\beta.$$

The elements  $X_1, \dots, X_m$  pairwise commute, so ordering of the factors in (2.8) corresponding to the indices  $k$  does not matter.

Now for each  $l = 1, \dots, n$  introduce the element of the algebra  $\text{End}(\mathbb{C}^N)^{\otimes(m+n)}$

All the elements  $X_1, \dots, X_{m+n}$  also pairwise commute. Let us use the properties of the element  $Y_\beta \in \text{End}(\mathbb{C}^N)^{\otimes n}$  similar to (2.7):

$$(P_{1l} + \dots + P_{l-1,l}) \cdot Y_\beta = b_l Y_\beta; \quad l = 1, \dots, n.$$

Any symmetric polynomial in the sums  $P_{12} + \dots + P_{1l}$  with  $l = 1, \dots, n$  belongs to the image in  $\text{End}(\mathbb{C}^N)^{\otimes n}$  of the centre of the group ring  $\mathbb{C} \cdot S_n$ . Therefore

$$\prod_l (h - P_{l,l+1} - \dots - P_{ln} - z) \cdot Y_\beta = \prod_l (h - P_{1l} - \dots - P_{l-1,l} - z) \cdot Y_\beta = \prod_l (z - b_l - h) \cdot Y_\beta$$

in  $\text{End}(\mathbb{C}^N)^{\otimes n}$ . Hence we can rewrite the right hand side of the equality (2.8) as

$$(2.9) \quad \prod_k \frac{h - X_k - z}{h + a_k - z} \cdot \prod_l \frac{h - X_{m+l} - z}{h - b_l - z} \cdot Y_\alpha \otimes Y_\beta.$$

Note that for any  $h, z \in \mathbb{C}$  the operator  $(h - X_1 - z) \dots (h - X_{m+n} - z)$  is the image in  $\text{End}(\mathbb{C}^N)^{\otimes(m+n)}$  of a central element in the group ring  $\mathbb{C} \cdot S_{m+n}$ . The eigenvalue of that central element in the irreducible  $S_{m+n}$ -module corresponding to the Young diagram with the contents  $c_1, \dots, c_{m+n}$  equals  $(h - c_1 - z) \dots (h - c_{m+n} - z)$ .

The image of the operator  $Y_\alpha \otimes Y_\beta$  in  $(\mathbb{C}^N)^{\otimes(m+n)}$  is the tensor product of the irreducible  $\mathfrak{gl}_N$ -modules of highest weights  $(\alpha_1, \dots, \alpha_N)$  and  $(\beta_1, \dots, \beta_N)$ . Let  $\gamma = (\gamma_1, \dots, \gamma_N)$  run through the set  $\Gamma_{\alpha\beta}$  of highest weights of all irreducible  $\mathfrak{gl}_N$ -modules appearing in this tensor product. We identify  $\gamma$  with corresponding usual Young diagram. The expression (2.9) for the product (2.5) shows that the operator  $R_{\alpha\beta}(h)$  is not invertible if and only if  $h$  is a content for at least one but not for all diagrams  $\gamma \in \Gamma_{\alpha\beta}$ . That is if and only if  $h \in \mathbb{Z}$  and

$$\min_{\gamma} \gamma_i < h + i \leq \max_{\gamma} \gamma_i$$

for some  $i \in \{1, \dots, N\}$ . Let the index  $i$  be fixed. Theorem 2.3 will follow from

$$(2.10) \quad \min_{\gamma} \gamma_i = \max(\alpha_i + \beta_N, \alpha_{i+1} + \beta_{N-1}, \dots, \alpha_N + \beta_i),$$

$$(2.11) \quad \max_{\gamma} \gamma_i = \min(\alpha_1 + \beta_i, \alpha_2 + \beta_{i-1}, \dots, \alpha_i + \beta_1).$$

First let us check that the numbers  $\gamma_i$  with the fixed index  $i$  attain the values at the right hand sides of (2.10) and (2.11). We will use the following well-known fact [B, Exercice VIII.9.14]: the unique dominant weight in the  $S_N$ -orbit of the weight  $(\alpha_1 + \beta_N, \alpha_2 + \beta_{N-1}, \dots, \alpha_N + \beta_1)$  belongs to  $\Gamma_{\alpha\beta}$ . Apply this fact to  $(\alpha_i, \dots, \alpha_N)$  and  $(\beta_i, \dots, \beta_N)$  instead of  $(\alpha_1, \dots, \alpha_N)$  and  $(\beta_1, \dots, \beta_N)$ . Then with the help of Littlewood-Richardson rule [M, Theorem I.9.2] one obtains that the set  $\Gamma_{\alpha\beta}$  contains the weight  $\gamma$  such that  $\gamma_j = \alpha_j + \beta_j$  for  $j < i$  while

$$\gamma_i = \max(\alpha_i + \beta_N, \dots, \alpha_N + \beta_i), \quad \dots, \quad \gamma_N = \max(\alpha_i + \beta_N, \dots, \alpha_N + \beta_i).$$

So the value at the right hand side of (2.10) is attained by  $\gamma_i$ . Similarly, by applying that well-known fact to  $(\alpha_1, \dots, \alpha_i)$  and  $(\beta_1, \dots, \beta_i)$  one gets  $\gamma \in \Gamma_{\alpha\beta}$  where

$$\gamma_1 = \max(\alpha_1 + \beta_i, \dots, \alpha_i + \beta_1), \quad \dots, \quad \gamma_i = \min(\alpha_1 + \beta_i, \dots, \alpha_i + \beta_1)$$

and  $\gamma_j = \alpha_j + \beta_j$  for  $j > i$ . So  $\gamma_i$  attains the value at the right hand side of (2.11)

It now remains to prove for the fixed indices  $i, j \in \{1, \dots, N\}$  the inequalities

$$(2.12) \quad \gamma_i \geq \alpha_{N-j+i} + \beta_j \quad \text{if} \quad i \leq j,$$

$$(2.13) \quad \gamma_i \leq \alpha_{i-j+1} + \beta_j \quad \text{if} \quad i \geq j.$$

These inequalities follow easily from the Littlewood-Richardson rule. Recall that the diagram  $\gamma$  occurs in  $\Gamma_{\alpha\beta}$  if there exists a semi-standard tableau  $\kappa$  of shape  $\gamma/\beta$  such that the values  $1, \dots, N$  are taken by  $\kappa$  respectively  $\alpha_1, \dots, \alpha_N$  times and the tableau  $\kappa$  satisfies the lattice property [M, Section I.9]. Fix such a tableau  $\kappa$  and for  $j, k = 1, \dots, N$  denote by  $\alpha_{jk}$  the number of times  $\kappa$  takes the value  $k$  in the row  $j$ . Then  $\alpha_{jk} = 0$  for  $j < k$  since  $\kappa$  is semi-standard and has the lattice property. Let us check inequality (2.12). Write  $l = N - j + i$  for short. If  $i \leq j$

$$\alpha_l = \alpha_{ll} + \dots + \alpha_{Nl} \leq \alpha_{l-1, l-1} + \dots + \alpha_{N-1, l-1} \leq \dots \leq \alpha_{ii} + \dots + \alpha_{ji}$$

because of the lattice property. Further, here we have  $\alpha_{ii} + \dots + \alpha_{ji} \leq \gamma_i - \beta_j$  since  $\kappa$  is semi-standard. Thus we obtain (2.12).

Now suppose that  $i \geq j$ . Let us write  $l = i - j + 1$  for short. Then we have  $\gamma_i - \beta_j \leq \alpha_{ii} + \dots + \alpha_{il}$  because  $\kappa$  is semi-standard. Here by the lattice property

$$\begin{aligned} \alpha_{ii} + \dots + \alpha_{il} &\leq \alpha_{i-1, i-1} + \alpha_{i, i-1} + \dots + \alpha_{il} \leq \\ &\alpha_{i-2, i-2} + \alpha_{i-1, i-2} + \alpha_{i, i-2} + \dots + \alpha_{il} \leq \dots \leq \alpha_{ll} + \dots + \alpha_{il} \leq \alpha_l. \end{aligned}$$

Thus we obtain the inequality (2.13) and complete the proof of Theorem 2.3  $\square$

### 3. Cyclicity conditions

In this section we will consider the tensor product of  $n$  elementary  $Y(\mathfrak{gl}_N)$ -modules for any  $n$ . For each  $s = 1, \dots, n$  fix a non-negative integer  $M^{(s)}$  and take a pair of non-increasing sequences of integers  $\lambda^{(s)}, \mu^{(s)}$  with lengths  $N + M^{(s)}, M^{(s)}$  respectively. Take a parameter  $h^{(s)} \in \mathbb{C}$ . Consider the elementary  $Y(\mathfrak{gl}_N)$ -module  $V^{(s)} = V_{\lambda^{(s)}, \mu^{(s)}}(h^{(s)})$ . Its Drinfeld polynomials  $P_1^{(s)}(u), \dots, P_{N-1}^{(s)}(u)$  are given explicitly by Proposition 1.6. Introduce the rational functions

$$(3.1) \quad Q_k^{(s)}(u) = P_k^{(s)}(u) / P_k^{(s)}(u+1); \quad k = 1, \dots, N-1.$$

Further, denote by  $\mathcal{X}_k^{(s)}$  the collection of all numbers of the form

$$(3.2) \quad i - h^{(s)} - \lambda_{M^{(s)}+k, i} - 1; \quad i = 1, \dots, M^{(s)} + k$$

where  $\lambda_{M^{(s)}+k, i}$  is the  $(M^{(s)} + k, i)$ -entry of any scheme  $\Lambda$  in  $\mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}$  such that the array  $\Lambda^-$  obtained from  $\Lambda$  by decreasing this entry by 1 is also in  $\mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}$ .

Let  $\zeta^{(s)} \in V^{(s)}$  be a singular vector, it is determined up to scalar multiplier. Consider  $Y(\mathfrak{gl}_N)$ -module  $V$  obtained from the tensor product  $V^{(1)} \otimes \dots \otimes V^{(n)}$  via the comultiplication (1.18). Consider the vector  $\zeta = \zeta^{(1)} \otimes \dots \otimes \zeta^{(n)} \in V$ . By Proposition 1.8 for any  $k = 1, \dots, N-1$  the coproduct  $\Delta^{(n)}(C_k(u))$  is a sum of the elements in  $Y(\mathfrak{gl}_N)^{\otimes n}[[u^{-1}]]$  with the degrees in  $\mathbb{Z}^n$  containing at least one positive component. Therefore  $C_k(u) \cdot \zeta = 0$  in the module  $V$  for any index  $k$ . The next proposition gives sufficient conditions for cyclicity of the vector  $\zeta$  under the action of the coefficients of the series  $P_k^{(s)}(u) = \sum_{i=0}^{\infty} p_{k,i}^{(s)} u^i$  in the module  $V$ .

**Proposition 3.1.** *Suppose that  $Q_k^{(s)}(x) \neq 0$  for any  $x \in \mathcal{X}_k^{(r)}$  when  $1 \leq k < n$  and  $1 \leq r < s \leq n$ . Then the vector  $\zeta$  in the  $Y(\mathfrak{gl}_N)$ -module  $V$  is cyclic.*

*Proof.* Each of the vector spaces  $V_{\lambda^{(s)}, \mu^{(s)}}$  has a  $\mathbb{Z}$ -grading as defined in the end of Section 1. Thus the space  $V$  acquires grading by the elements of the group  $\mathbb{Z}^n$ . We will equip the set  $\mathbb{Z}^n$  with lexicographical ordering: the element  $(d_1, \dots, d_n)$  precedes  $(d'_1, \dots, d'_n)$  if  $d_s < d'_s$  for some index  $s$  while  $d_r = d'_r$  for each  $r < s$ .

Take any vector  $\xi = \xi^{(1)} \otimes \dots \otimes \xi^{(n)} \in V$  where any tensor factor  $\xi^{(s)}$  is an element of the Gelfand-Zetlin basis in  $V_{\lambda^{(s)}, \mu^{(s)}}$ . Fix any index  $r \in \{1, \dots, n\}$  and assume that  $\xi^{(s)} = \zeta^{(s)}$  for every  $s > r$ . Write  $\xi^{(r)} = \xi_\Lambda$  for a certain scheme  $\Lambda$  in  $\mathcal{S}_{\lambda^{(r)}, \mu^{(r)}}$ . Then fix any indices  $k \in \{1, \dots, N-1\}$  and  $i \in \{1, \dots, M^{(r)} + k\}$  such that the array  $\Lambda^-$  obtained from  $\Lambda$  by decreasing the  $(M^{(r)} + k, i)$ -entry by 1, is again in  $\mathcal{S}_{\lambda^{(r)}, \mu^{(r)}}$ . We have  $x = i - h^{(r)} - \lambda_{M^{(r)}+k, i} - 1 \in \mathcal{X}_k^{(r)}$ . Take

$$(3.3) \quad \xi^{(1)} \otimes \dots \otimes \xi^{(r-1)} \otimes \xi_{\Lambda^-} \otimes \zeta^{(r+1)} \otimes \dots \otimes \zeta^{(n)} \in V.$$

Consider the rational function  $B_k(u) \cdot \xi$  of  $u$  valued in  $V$ . Take the term of this function with the leading degree in  $\mathbb{Z}^n$ . It is again a rational function of  $u$  valued in  $V$ . Let  $\eta \in V$  be the first non-zero coefficient of the Laurent series in  $u - x$  of the latter function. Here and in what follows we consider the Laurent expansions at  $u = x$ . We will show that the vector  $\eta$  is a scalar multiple of (3.3). This guarantees the cyclicity of the vector  $\zeta = \zeta^{(1)} \otimes \dots \otimes \zeta^{(n)}$  under the action of the coefficients of the series  $B_1(u), \dots, B_{N-1}(u)$  in the module  $V$ .

Observe that by Proposition 1.8 the coproduct  $\Delta(B_k(u))$  is equal to the sum

$$A_k(u) \otimes B_k(u) + B_k(u) \otimes D_k(u)$$

plus the terms of degrees in  $\mathbb{Z}^2$  with a positive second component. Therefore by our assumption on  $\xi$  the vector  $B_k(u) \cdot \xi \in V^{(1)} \otimes \dots \otimes V^{(n)}$  equals the sum

$$(3.4) \quad (A_k(u) \cdot \xi^{(1)} \otimes \dots \otimes \xi^{(r)}) \otimes (B_k(u) \cdot \zeta^{(r+1)} \otimes \dots \otimes \zeta^{(n)}) + \\ (B_k(u) \cdot \xi^{(1)} \otimes \dots \otimes \xi^{(r)}) \otimes (D_k(u) \cdot \zeta^{(r+1)} \otimes \dots \otimes \zeta^{(n)}),$$

where the actions of the algebra  $Y(\mathfrak{gl}_N)$  in  $V^{(1)} \otimes \dots \otimes V^{(r)}$  and  $V^{(r+1)} \otimes \dots \otimes V^{(n)}$  are determined via the comultiplications  $\Delta^{(r)}$  and  $\Delta^{(n-r)}$  respectively.

By the second equality in (1.7) the series  $D_k(u)$  is equal to the sum of

$$T_{k+1, k+1}(u - k + 1) \cdot T_{k-1, k-1}(u - k + 2) \dots T_{22}(u - 1) T_{11}(u)$$

and certain elements from the ideal  $J$ , see Section 1. The series  $B_k(u)$  is the sum of

$$T_{k, k+1}(u - k + 1) \cdot T_{k-1, k-1}(u - k + 2) \dots T_{22}(u - 1) T_{11}(u)$$

and again of certain elements from  $J$ . But the vector  $\zeta^{(r+1)} \otimes \dots \otimes \zeta^{(n)}$  is an eigenvector for the action in  $V^{(r+1)} \otimes \dots \otimes V^{(n)}$  of the product

Hence by dividing the vector (3.4) by the corresponding eigenvalue we get the sum

$$(3.5) \quad (A_k(u) \cdot \xi^{(1)} \otimes \dots \otimes \xi^{(r)}) \otimes (T_{k,k+1}(u - k + 1) \cdot \zeta^{(r+1)} \otimes \dots \otimes \zeta^{(n)}) + \\ (B_k(u) \cdot \xi^{(1)} \otimes \dots \otimes \xi^{(r)}) \otimes (T_{k+1,k+1}(u - k + 1) \cdot \zeta^{(r+1)} \otimes \dots \otimes \zeta^{(n)}).$$

Note that by taking here the first non-zero Laurent coefficient at  $u = x$  of the component with the leading degree in  $\mathbb{Z}^n$ , we get a scalar multiple of the same vector  $\eta$  as in  $B_k(u) \cdot \xi$ .

Further, by Proposition 1.8 the components of  $\Delta^{(r)}(A_k(u))$  and  $\Delta^{(r)}(B_k(u))$  with the leading degrees in  $\mathbb{Z}^n$  are respectively

$$A_k(u)^{\otimes(r-1)} \otimes A_k(u) \quad \text{and} \quad A_k(u)^{\otimes(r-1)} \otimes B_k(u).$$

But by Theorem 1.3 the tensor product  $(A_k(u) \cdot \xi^{(1)}) \otimes \dots \otimes (A_k(u) \cdot \xi^{(r-1)})$  equals  $\xi^{(1)} \otimes \dots \otimes \xi^{(r-1)}$  times a certain rational function of  $u$  valued in  $\mathbb{C}$ . Divide (3.5) by this rational function. Now it suffices to show that by taking in

$$(3.6) \quad (A_k(u) \cdot \xi_\Lambda) \otimes (T_{k,k+1}(u - k + 1) \cdot \zeta^{(r+1)} \otimes \dots \otimes \zeta^{(n)}) + \\ (B_k(u) \cdot \xi_\Lambda) \otimes (T_{k+1,k+1}(u - k + 1) \cdot \zeta^{(r+1)} \otimes \dots \otimes \zeta^{(n)})$$

the first non-zero Laurent coefficient at  $u = x$ , we get scalar multiple of the vector

$$\xi_{\Lambda^-} \otimes \zeta^{(r+1)} \otimes \dots \otimes \zeta^{(n)}.$$

Let  $R_{k+1}^{(r+1)}(u), \dots, R_{k+1}^{(n)}(u)$  be eigenvalues of  $T_{k+1,k+1}(u)$  on  $\zeta^{(r+1)}, \dots, \zeta^{(n)}$  respectively. By (1.16) coproduct  $\Delta^{(n-r)}(T_{k+1,k+1}(u))$  equals  $T_{k+1,k+1}(u)^{\otimes(n-r)}$  plus terms with the degrees in  $\mathbb{Z}^{n-r}$  containing at least one positive component. So

$$T_{k+1,k+1}(u) \cdot \zeta^{(r+1)} \otimes \dots \otimes \zeta^{(n)} = R_{k+1}^{(r+1)}(u) \dots R_{k+1}^{(n)}(u) \cdot \zeta^{(r+1)} \otimes \dots \otimes \zeta^{(n)}.$$

Determine the rational function  $\rho_k(u)$  by (1.9) for  $M = M^{(r)}, h = h^{(r)}, \mu_i = \mu_i^{(r)}$ . By Theorem 1.3 the value of  $\rho_k(u) A_k(u) \cdot \xi_\Lambda$  at  $u = x$  is zero. By Theorem 1.4 the value of  $\rho_k(u) B_k(u) \cdot \xi_\Lambda$  at  $u = x$  is a non-zero scalar multiple of vector  $\xi_{\Lambda^-}$ . Divide (3.6) by

$$R_{k+1}^{(r+1)}(u - k + 1) \dots R_{k+1}^{(n)}(u - k + 1).$$

It now remains to prove regularity at  $u = x$  of the rational function

$$(3.7) \quad \frac{T_{k,k+1}(u - k + 1) \cdot \zeta^{(r+1)} \otimes \dots \otimes \zeta^{(n)}}{R_{k+1}^{(r+1)}(u - k + 1) \dots R_{k+1}^{(n)}(u - k + 1)}.$$

Thanks to the equalities (1.15) we get from (3.1) that for every  $s = r + 1, \dots, n$

$$(3.8) \quad Q^{(s)}(u) = P^{(s)}(u - k + 1) / P^{(s)}(u - k + 1)$$

On the other hand, the coproduct  $\Delta^{(n-r)}(T_{k,k+1}(u))$  has the form

$$\sum_{s>r} T_{kk}(u)^{\otimes(s-r-1)} \otimes T_{k,k+1}(u) \otimes T_{k+1,k+1}(u)^{\otimes(n-s)}$$

plus terms with the degrees in  $\mathbb{Z}^{n-r}$  containing at least one positive component. So due to (3.8) the vector (3.7) is equal to the sum over  $s = r+1, \dots, n$  of the vectors

$$\frac{\zeta^{(r+1)} \otimes \dots \otimes \zeta^{(s-1)}}{Q_k^{(r+1)}(u) \dots Q_k^{(s-1)}(u)} \otimes \frac{T_{k,k+1}(u-k+1) \cdot \zeta^{(s)}}{R_{k+1}^{(s)}(u-k+1)} \otimes \zeta^{(s+1)} \otimes \dots \otimes \zeta^{(n)}.$$

Here  $Q_k^{(r+1)}(x), \dots, Q_k^{(s-1)}(x) \neq 0$  by our assumption. Thus to complete the proof it suffices to show that the rational function in  $u$

$$(3.9) \quad T_{k,k+1}(u-k+1) \cdot \zeta^{(s)} / R_{k+1}^{(s)}(u-k+1)$$

is also regular at  $u = x$ . Suppose this function is not identically zero. Denote by  $\varpi$  the first non-zero Laurent coefficient of this function at  $u = x$ . Let  $v$  be a formal parameter. The  $Y(\mathfrak{gl}_N)$ -module  $V^{(s)}$  is irreducible and its subspace of maximal  $\mathbb{Z}$ -degree is spanned by vector  $\zeta^{(s)}$ . Therefore  $T_{l+1,l}(v) \cdot \varpi \neq 0$  as a formal series in  $v$  for at least one index  $l \in \{1, \dots, N-1\}$ . But

$$(3.10) \quad T_{l+1,l}(v) T_{k,k+1}(u) \cdot \zeta^{(s)} = \frac{T_{kl}(v) T_{l+1,k+1}(u) - T_{kl}(u) T_{l+1,k+1}(v)}{u-v} \cdot \zeta^{(s)}$$

due to the defining relations (1.2). Hence  $T_{l+1,l}(v) \cdot \varpi = 0$  for any index  $l > k$ . If  $l < k$  then by applying (1.2) to the right hand side of the equality (3.10), we get  $T_{l+1,l}(v) \cdot \varpi = 0$  again. Thus the series  $T_{k+1,k}(v) \cdot \varpi$  in  $v$  is not identically zero. On the other hand, by applying (3.10) to  $l = k$  we get the equality

$$T_{k+1,k}(v) \cdot \frac{T_{k,k+1}(u-k+1)}{R_{k+1}^{(s)}(u-k+1)} \cdot \zeta^{(s)} = \frac{1}{u-v-k+1} \left( R_k^{(s)}(v) - \frac{R_{k+1}^{(s)}(v)}{Q_k^{(s)}(u)} \right) \cdot \zeta^{(s)}$$

where the right hand has no pole at  $u = x$  because  $Q_k^{(s)}(x) \neq 0$  by our assumption. Therefore the rational function (3.9) is indeed regular at  $u = x$   $\square$

The next proposition matches Proposition 3.1 and is essentially equivalent to it.

**Proposition 3.2.** *Suppose that  $Q_k^{(s)}(x) \neq 0$  for any  $x \in \mathcal{X}_k^{(r)}$  when  $1 \leq k < n$  and  $1 \leq s < r \leq n$ . Then the vector  $\zeta$  in the  $Y(\mathfrak{gl}_N)$ -module  $V$  is cocyclic.*

*Proof.* Let us consider the  $Y(\mathfrak{gl}_N)$ -module dual to  $V$ . Due to Proposition 1.7 and to (1.17) it is equivalent to the  $Y(\mathfrak{gl}_N)$ -module obtained from the tensor product  $V^{(1)} \otimes \dots \otimes V^{(n)}$  by composing the comultiplication (1.18) with the transposition  $(1, \dots, n) \mapsto (n, \dots, 1)$  of the tensor factors. Denote by  $V'$  the latter module. The cocyclicity of the vector  $\zeta = \zeta^{(1)} \otimes \dots \otimes \zeta^{(n)}$  in  $V$  amounts to the cyclicity of the same vector in  $V'$ . Now Proposition 3.1 provides the required statement  $\square$

By combining Propositions 3.1 and 3.2 we immediately obtain sufficient conditions for irreducibility of the  $Y(\mathfrak{gl}_N)$ -module  $V = V^{(1)} \otimes \dots \otimes V^{(n)}$

**Theorem 3.3.** *Suppose that  $Q_k^{(s)}(x) \neq 0$  for any  $x \in \mathcal{X}_k^{(r)}$  whenever  $1 \leq k < n$  and  $r \neq s$ . Then the  $Y(\mathfrak{gl}_N)$ -module  $V$  is irreducible.*

In general, these conditions are not necessary for the irreducibility of  $V$ . Still by using again Proposition 3.1 we can give a criterion for the irreducibility of  $V$  when each of the skew Young diagrams  $\lambda^{(1)}/\mu^{(1)}, \dots, \lambda^{(n)}/\mu^{(n)}$  has the simplest shape.

First let us make a general remark. For  $s = 1, \dots, n$  take the  $Y(\mathfrak{gl}_N)$ -module  $V_{\lambda^{(s)}/\mu^{(s)}}(h)$  as defined in Section 2. Due to Proposition 2.1 the  $Y(\mathfrak{gl}_N)$ -module  $V^{(s)}$  can be obtained from it by pulling back through an automorphism of the form  $\omega_f$ . Let us fix this realization of  $V^{(s)}$ . Note that by the definition (1.16) we have the equalities

$$\Delta \circ \omega_f = (\omega_f \otimes \text{id}) \circ \Delta = (\text{id} \otimes \omega_f) \circ \Delta.$$

Therefore for any  $r < s$  the element

$$(3.11) \quad R_{\lambda^{(r)}/\mu^{(r)}, \lambda^{(s)}/\mu^{(s)}}(h^{(r)} - h^{(s)}) \in \text{End}(\text{im } Y_{\lambda^{(r)}/\mu^{(r)}} \otimes \text{im } Y_{\lambda^{(s)}/\mu^{(s)}})$$

is an intertwining operator between the  $Y(\mathfrak{gl}_N)$ -modules obtained from the tensor product  $V^{(r)} \otimes V^{(s)}$  via the comultiplications  $\Delta'$  and  $\Delta$  respectively.

**Theorem 3.4.** *Suppose that each of the skew diagrams  $\lambda^{(1)}/\mu^{(1)}, \dots, \lambda^{(n)}/\mu^{(n)}$  has rectangular shape. Then the  $Y(\mathfrak{gl}_N)$ -module  $V$  is irreducible if and only if the operator (3.11) is invertible whenever  $1 \leq r < s \leq n$ .*

*Proof.* If at least one of the operators (3.11) with  $r < s$  is not invertible then the  $Y(\mathfrak{gl}_N)$ -module  $V$  is reducible thanks to the general remark made above. Suppose that each of the operators (3.11) with  $r < s$  is invertible. We will show that then the  $Y(\mathfrak{gl}_N)$ -module  $V$  is irreducible. Without any loss of generality we can assume that

$$V^{(s)} = V_{\lambda^{(s)}, \mu^{(s)}}(h^{(s)}) = V_{\lambda^{(s)}/\mu^{(s)}}(h^{(s)})$$

for each  $s = 1, \dots, n$ . Recall that if a skew diagram  $\alpha$  is obtained by adding the same number  $c$  to every contents of a skew diagram  $\beta$  then  $V_\alpha(h) = V_\beta(h + c)$ . So we can further assume that for each  $s = 1, \dots, n$

$$(3.12) \quad \lambda^{(s)} = (k^{(s)}, \dots, k^{(s)}, 0, \dots, 0) \quad \text{and} \quad \mu^{(s)} = \emptyset$$

where the positive integer  $k^{(s)}$  appears  $l^{(s)} \leq N$  times. If  $l^{(s)}$  equals 0 or  $N$  then the  $Y(\mathfrak{gl}_N)$ -module  $V^{(s)}$  is one-dimensional. We will assume that  $0 < l^{(s)} < N$ .

If for some  $k \in \{1, \dots, N-1\}$  the number (3.2) appears in the set  $\mathcal{X}_k^{(s)}$  then

$$\max(0, l^{(s)} - N + k) < i \leq \min(k, l^{(s)})$$

and in this case the integer  $\lambda_{M^{(s)}+k, i} = \lambda_{ki}$  can vary from 1 up to  $k^{(s)}$ . Therefore

$$\mathcal{X}_k^{(s)} = \{h - h^{(s)} - 1 \mid \max(0, l^{(s)} - N + k) - k^{(s)} < h < \min(k, l^{(s)}), h \in \mathbb{Z}\}.$$

By Proposition 1.6 we get  $P_k^{(s)}(u) = Q_k^{(s)}(u) = 1$  for any  $k \neq l^{(s)}$ . If  $k = l^{(s)}$  then

$$P_k^{(s)}(u) = (u + h^{(s)} - l^{(s)} + 1) \dots (u + h^{(s)} - l^{(s)} + k^{(s)})$$

and the rational function  $Q_k^{(s)}(u)$  has the only zero  $u = l^{(s)} - h^{(s)} - 1$ .

Let us fix any indices  $r < s$ . Then  $Q_k^{(s)}(u)$  has a zero in  $\mathcal{X}_k^{(r)}$  only for  $k = l^{(s)}$  and only when we have the inequalities

$$(3.13) \quad -\min(l^{(s)}, N - l^{(r)}) - k^{(r)} < h^{(r)} - h^{(s)} < \min(0, l^{(r)} - l^{(s)})$$

while  $h^{(r)} - h^{(s)} \in \mathbb{Z}$ . By exchanging the triples  $(h^{(r)}, k^{(r)}, l^{(r)})$  and  $(h^{(s)}, k^{(s)}, l^{(s)})$  in (3.13) we obtain the inequalities

$$(3.14) \quad \max(0, l^{(r)} - l^{(s)}) < h^{(r)} - h^{(s)} < \min(l^{(r)}, N - l^{(s)}) + k^{(s)}$$

where again  $h^{(r)} - h^{(s)} \in \mathbb{Z}$ . Now observe that the inequalities (3.13) and (3.14) exclude each other. On the other hand, the  $Y(\mathfrak{gl}_N)$ -modules  $V^{(r)} \otimes V^{(s)}$  and  $V^{(s)} \otimes V^{(r)}$  obtained via the comultiplication  $\Delta$  are equivalent: composition of the exchange map  $V^{(s)} \otimes V^{(r)} \rightarrow V^{(r)} \otimes V^{(s)}$  with (3.11) is invertible and commutes with the action of  $Y(\mathfrak{gl}_N)$ . So we can assume that  $Q_k^{(s)}(x) \neq 0$  for any  $x \in \mathcal{X}_k^{(r)}$  and  $k \in \{1, \dots, n-1\}$ . Then the vector  $\zeta \in V$  is cyclic by Proposition 3.1.

The  $Y(\mathfrak{gl}_N)$ -module  $V'$  introduced in the proof of Corollary 3.2 is equivalent to  $V$ . The isomorphism  $V' \rightarrow V$  is given by composition of the operators (3.11) with

$$(r, s) = (1, 2), (1, 3), (2, 3), \dots, \dots, \dots, (1, n), \dots, (n-1, n).$$

This isomorphism preserves one-dimensional subspace in  $V$  spanned by vector  $\zeta$ . Indeed, each tensor factor  $\zeta^{(s)} \in V^{(s)}$  of  $\zeta = \zeta^{(1)} \otimes \dots \otimes \zeta^{(n)}$  has the maximal degree with respect to  $\mathbb{Z}$ -grading by eigenvalues of the action in  $V^{(s)}$  of the element

$$N E_{11} + (N-1) E_{22} + \dots + E_{NN} \in \mathfrak{gl}_N \subset Y(\mathfrak{gl}_N).$$

But the operator (3.11) commutes with the action of Lie algebra  $\mathfrak{gl}_N$  in  $V^{(r)} \otimes V^{(s)}$ . Thus cyclicity of the vector  $\zeta$  in the module  $V$  is equivalent to its cocyclicity  $\square$

When the diagrams  $\lambda^{(1)}/\mu^{(1)}, \dots, \lambda^{(n)}/\mu^{(n)}$  have rectangular shapes, Theorem 2.3 explicitly describes for each  $r < s$  the set of all points  $h^{(r)} - h^{(s)} \in \mathbb{Z}$  where the operator (3.11) is not invertible. Under the assumptions (3.12) the non-invertibility occurs if and only if one of the next two pairs of inequalities holds:

$$\begin{aligned} & -\min(l^{(s)}, N - l^{(r)}) - k^{(r)} < h^{(r)} - h^{(s)} < \min(0, l^{(r)} - l^{(s)}) + \min(0, k^{(s)} - k^{(r)}), \\ & \max(0, l^{(r)} - l^{(s)}) + \max(0, k^{(s)} - k^{(r)}) < h^{(r)} - h^{(s)} < \min(l^{(r)}, N - l^{(s)}) + k^{(s)}. \end{aligned}$$

Note that if  $k^{(r)} = k^{(s)}$  then these pairs coincide with (3.13) and (3.14) respectively. Therefore if  $k^{(1)} = \dots = k^{(n)}$  then already the conditions of Theorem 3.3 are necessary and sufficient for the irreducibility of the  $Y(\mathfrak{gl}_N)$ -module  $V$ . In the case  $k^{(1)} = \dots = k^{(n)} = 1$  our Theorem 3.4 follows from [AK, Theorem 4.1]. In the other special case  $l^{(1)} = \dots = l^{(n)} = 1$ , Theorem 3.4 follows from [Z, Theorem 4.2].

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